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Is memory in the minority game irrelevant?

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By analyzing the frequencies of selecting entries in the strategies and time series data of the minority game, we show that the memory in the minority game is irrelevant for the emergence of coordinate behavior. It is found that the memory in the game just yields periodic structures and statistically does not play any significant role as far as the volatility is concerned.

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The minority game [1] was introduced as a model for an adaptive system of interacting agents, motivated in part by Arthur's ''El Farol'' problem [2]. This model is interesting and important in the biological and social science, not to mention in the dynamics of the complex systems, in that the agents show coordinate behavior by adjusting their strategies adaptively to the environment.

The minority game consists of N agents, each of which has s strategies to make a decision, i.e., to play a game. At each time step of the game, agents are asked to join one of two groups, labeled 0 or 1, and those agents in the minority group win. When agents make decision to join a group, they select among their s strategies the most successful one up to that moment and use it to choose a group.

A strategy of size m is nothing but a table, which consists of 2^m rows (or entries) and two columns, ''history'' and ''prediction.'' The history column is composed of all of the 2^m possible combinations of m strings of 0's and 1's, and the prediction column is made up of a randomly chosen 0 or 1. When the most successful strategy is chosen by each agent, he/she observes which groups were the minority groups during the immediately past m time steps (past history of the winning groups). With this information, each agent finds the correct sequence of 0's and 1's in the history column of the strategy, and determine which group (0 or 1) to choose in the current time step by looking the corresponding prediction column.

There is another scheme for making the predictions [3]. In this scheme, at each time step, an entry in the strategy is chosen randomly, independent of the past history of the winning groups. That is, at each time step, the past history is just invented to play a role of a fake history. Thus, it does not require any information about the past winning groups. An advantage of this scheme is that one can have any positive number of entries, not necessarily restricted to 2^m. Therefore, the only difference between the two schemes is whether the *memory* of the past winning groups is taken into account. To distinguish between these two schemes, we call the former, ''with memory,'' and the latter, ''without memory.''

In the minority game, one is especially interested in the standard deviation σ (*volatility* in the financial term) of the number of agents of a particular group, say group 0. That is,

$$\sigma = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(n(t)-\frac{N}{2}\right)^2}\ , \tag{1}$$

where N is the number of agents and $n(t)$ is the number of agents in the particular group at time step t. The volatility measures how effective the system is in terms of minimizing the risk: the smaller σ is, the larger the number of winners there are. In particular, if all agents do not have any strategies and just play the game randomly, the volatility would be simply $\sigma_0 = \sqrt{N/4}$. A remarkable feature of this model, however, is the adaptiveness and the emergence properties due to a coordinate behavior among agents. Without any centralized control mechanism acting on the agents, this model shows coordination among the agents. It is found in [4] that σ depends on m, furthermore, there is a region of m in which the volatility σ is smaller than σ_0. These two findings indicate the significance of the role of the memory size m in the strategies.

There is, however, a controversial question as to whether or not the *memory* of the game is relevant. In particular, in [3] using without memory scheme, it was shown via an extensive simulation that the game was completely independent of the history of the winning groups. They discovered that the behavior of the volatility depended on the *size, not on the history*, of the memory in the strategies. Using without memory scheme, they obtained the same features of the game, especially m and N dependence of the volatility. On the other hand, in [5], the game was analyzed in terms of De Bruijn graphs and it was argued that all quantities of the model depend on real histories of the game. In addition to this, in [4], the phase transition of the model with memory was studied and it was argued that the phase transition was due to the competition between an embedded periodic dynamics and the emergent coordination among the agents' response. If the periodic dynamics does indeed play a crucial role in the phase of the model, the outcome of the game should depend on the history of the past winning groups.

The models for the complex adaptive system, in general, incorporate *ab initio* many mechanisms into the models, thus it is not easy to understand which factors in the models result in various observations. Therefore, it is important to clarify the role of the memory in the game, not only for sorting out the apparent discrepancy, but also for a better understanding of the characteristics of the model. It is the purpose of this Rapid Communication to disentangle the controversial ques-

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tion about the role of the memory in the model.

The assertion that the volatility is insensitive to the memory [3] is quite understandable, since the volatility is a statistical quantity that does not contain any temporal information: one can scramble the entire data by permuting all quantities randomly to generate a surrogate data that has the exactly the same volatility. Therefore, it is important to figure out how the history of the winning groups, or the memory, affects the outcome of the game.

To figure out the role of the memory in the game, we need to scrutinize how the agents choose a group to join. Note that the agents decide which group to join via entries in their strategies regardless of the history is taken into consideration. Thus, the volatility depends on how many times the agents use each entry to make decisions during the game. Therefore, a relevant quantity to be investigated is the frequency of selecting entries in both schemes.

To this end, we simulate the game with $m=2$ and focus on an entry $((0,0),*)$ [6], where the history column is (0, 0) and $*$ in the prediction column can be either 0 or 1. With this, we keep track of the number of time steps between successive selections of the same entry during the game.

In the case of without memory, an entry is chosen randomly with an equal probability. Thus, the total number of selecting a particular entry up to a certain time step is a Poisson process with the rate $\lambda=1/2^m$, where 2^m is the total number of entries. Therefore, the probability $P(t)$ of the number of time steps t between successive selections of the same entry is an exponential distribution with the same rate $\lambda=1/2^m$. This is shown in Fig. 1(a) by simulating the game with $m=2$ and $m=3$.

In the case of with memory, however, the probability distribution is not an exponential, but exhibits relatively higher probabilities at certain time steps. More specifically, as seen in Fig. 1(b), the probability distribution is symmetric with respect to $t=7$, except for $t=1$, and there are two peaks at $t_1=1$ and $t_2=7$ time steps [7]. This means that the agents are apt to select the same entry again at 1 or 7 time steps later.

From Fig. 1, we have found that the probability distribution of the time steps for selecting the same entries was different from which scheme was used. What we are interested in, however, is the frequency of selecting the entry during the game. Thus, we calculate the expectation values of the time step using the probability distributions for two schemes, $\langle T\rangle=\Sigma_t tP(t)$. For without memory case, the number of steps between successive selections of the entry follows an exponential distribution with the rate $\lambda=1/2^m$. Thus, the expectation value is just the inverse of the rate, i.e., $\langle T\rangle=2^m=4$, when $m=2$. This can be also confirmed by the simulation. In the case of with memory, we find from Fig. 1(b) that $\langle T\rangle=2^m=4$, when $m=2$, depending only on m.

From the above calculations, we find that the average value for both schemes are the same, even though their probability distributions are different. That is, on the average, the frequency of choosing an entry is the same whether the game utilizes the memory or not. This implies that the average number of agents' decisions to join one of two groups should

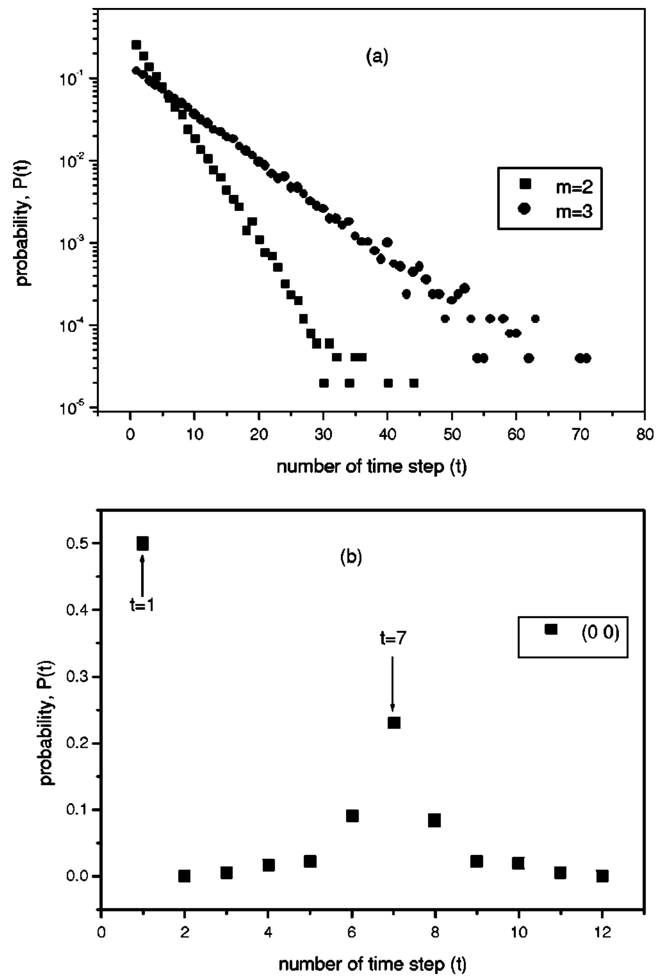

FIG. 1. Probabilities of the number of time steps for the same entry to be selected again, for $s=2$, $m=2$, and $N=101$. In Fig. 1(a) (top), the ordinate is logarithmic scales, and in Fig. 1(b) (bottom), the ''prediction'' column (0, 0) is chosen for the simulation.

be independent of the history of the past winning groups, thus yielding the *same* volatility.

While the above finding statistically explains the same volatility for both schemes, it still remains to understand the role of the memory in the game. To investigate this, we calculate the power spectrum of the time series, which is composed of the number of time steps between successive selections of the same entry. This data was also used to generate the probability distribution of Fig. 1(b). The result of the power spectrum is shown in Fig. 2, in which we normalized the frequencies f such that $0\leqslant f\leqslant 1$, so that the following relation holds:

$$t_p=\frac{1}{f}, \tag{2}$$

where t_p is the corresponding periodic time step.

Figure 2 shows that the power spectrum has a peak at $f=1/2$, thus periodicity of the time series $t_p=2$. Note also that the probability distribution [Fig. 1(b)] has two peaks at $t_1=1$ and $t_2=7$, and symmetric around $t_2=7$ excluding t_1

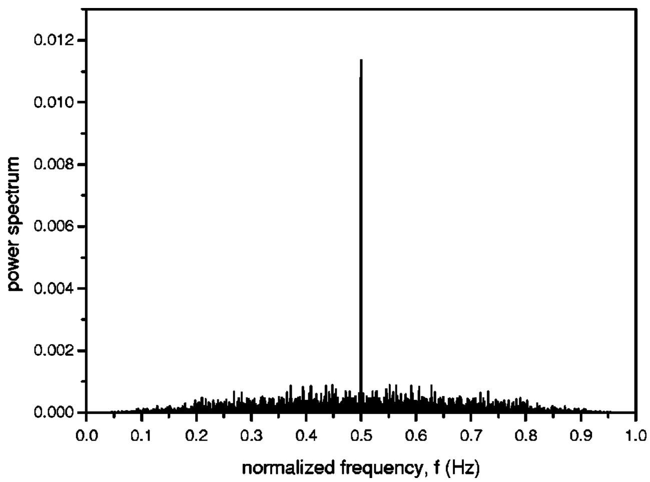

FIG. 2. Power spectrum of the time series of time steps between successive selection of (0, 0), with $s=2$, $m=2$, and $N=101$. Frequencies are normalized such that $0\leqslant f\leqslant 1$.

$=1$. These imply that the time steps for selecting the same entry *oscillate* between $t_1=1$ and $t_2=7$ (on the average sense), with the period of

$$t_1+t_2=2^{m+1}=8, \quad \text{for} \quad m=2. \tag{3}$$

This means that the periodicity is twice as many as the average number of time steps between successive selection. That is, the game with memory has a *periodic* structure at every $2^{m+1}=8$ time steps, when $m=2$. It can be shown that the period of the oscillation is independent of the choice of an entry, that is, not restricted to the prediction column (0, 0), and depends only on the size m.

To further support the above argument, we take two sets of time series, one is with memory and the other is without memory. Each set of data contains the number of agents in a particular group in each time step and we calculate the power spectra for both. In the case of without memory, power spectrum shows only the background noise or randomness, which implies that there is no periodicity, thus no temporal correlation, as expected. In the case of with memory, however, there are three peaks with distinguishable strengths as shown in Fig. 3.

As one can see from the Fig. 3, there are 3 peaks at $f_1=0.125$, $f_2=0.250$, and $f_3=0.375$, in the order of increasing frequency. With Eq. (2), we can calculate the corresponding periodicity of the fundamental frequency f_1 and find $t_p=1/f_1=8$ time steps, which is consistent with the result obtained from Eq. (3). Therefore, the periodic structure of the returning to the same entry founded above is responsible for the fundamental frequency f_1.

In addition to the fundamental frequency f_1, Fig. 3 shows the other two peaks in the higher frequency region. Moreover, they are harmonically related, that is, one is the sum of the other two: $f_2=f_1+f_1$ and $f_3=f_1+f_2$. To investigate this further, we calculate the bicoherence spectrum of the above time series. The bicoherence spectrum of a time series is a normalized bispectrum, which is an ensemble average of a

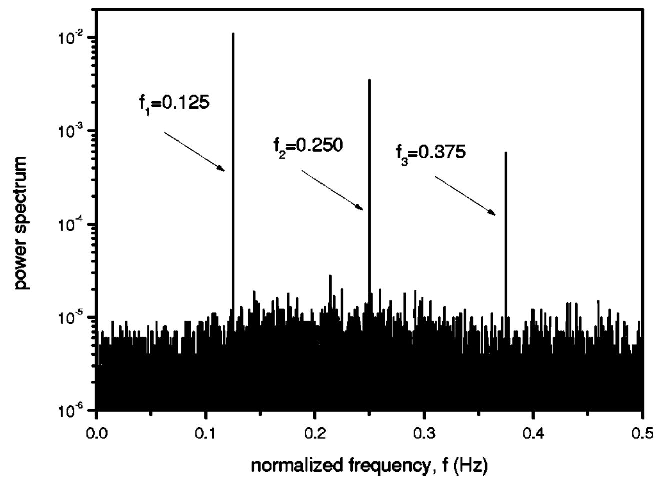

FIG. 3. Power spectrum of the time series of number of agents in a particular group, with $s=2$, $m=2$, and $N=101$. We normalized the frequencies f such that $0\leqslant f\leqslant 1$. Because of the symmetry, only f's for $0\leqslant f\leqslant 1/2$ are shown and the ordinate is logarithmic scale.

product of three spectral components. It is useful tool to discriminate between quadratically coupled signals, and the self-excited or spontaneously excited signals [8].

Figure 4 shows the result of the bicoherence spectrum estimator for the time series used for Fig. 3. From Fig. 4, one can see that there are two significant peaks at $(k,l)=(f_1,f_1)$ and $(k,l)=(f_1,f_2)=(f_2,f_1)$ with $\widehat{b^2}(f_1,f_1)=0.97$ and $\widehat{b^2}(f_1,f_2)=\widehat{b^2}(f_2,f_1)=0.91$. This means that 97% of the strength of f_2 is attributed to the quadratic self-excited phase coupling of f_1, and 91% of f_3, the quadratic coupling between f_1 and f_2. Once these effects are subtracted, these strengths are substantially reduced to the level of the background noise. This clearly explains that most of the power spectrum strength at f_2 and f_3 in Fig. 3 are due

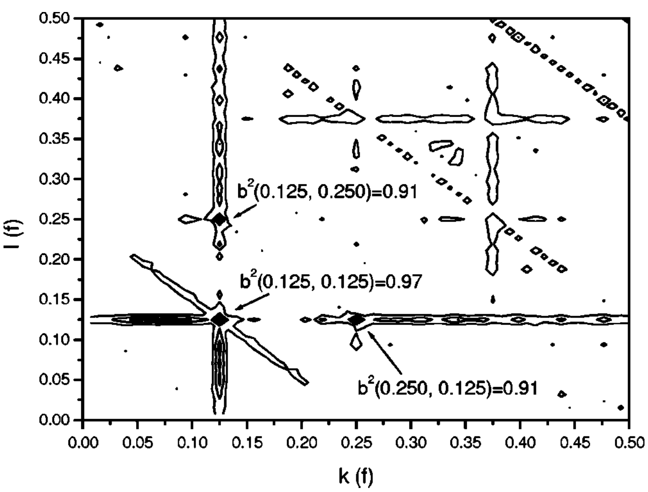

FIG. 4. Contour graph of the squared bicoherence spectrum of the time series of number of agents in a particular group, with $s=2$, $m=2$, and $N=101$. We take 32 sets of data, each of which in turn has 128 data.

to the harmonics and the nonlinear phase coupling of the fundamental frequency f_1.

From the above discussions, we have found that the difference between with memory and without memory is whether the periodic structure exists or not. For the model with memory, the model contains a deterministic nature *per se* through the history of the past winning groups. This is responsible for the periodic structure observed through the power spectrum and the bicoherence spectrum analyses. For the model without memory, on the other hand, there is no room for the past history, thus no periodicity. It is important to note that the deterministic nature, and thus the periodic structure, do not affect the volatility.

Since the quantity that we are interested in is the volatility, which is a statistical quantity insensitive of the temporal information such as a periodicity, we should obtain the same volatility as long as the other parameters remain the same. Therefore, the role of the memory is to generate a periodicity, which depends on the size of m.

In this discussion, we mainly concentrated on one set of parameters, $m=2$, $N=101$, and $s=2$. One can, however, certainly extend this and obtain consistent results. For example, when $m=3$, we can show that the periodicity is $t_p=2^{m+1}=16$, the fundamental frequency, $f_1=0.0625$, as well as the similar behavior of the quadratic couplings.

Throughout the study, we define the volatility as Eq. (1). One can, however, replace $N/2$ in Eq. (1) by $\langle n(t)\rangle_N$, an average number of agents in a particular group in each time step, and use it as the definition of the volatility. This definition renders to the case of the asymmetric phase [5]. It is, however, found that the periodic and/or nonperiodic structures do not depend on the definition of the volatility. This is understandable, since, for a small N relative to m, $\langle n(t)\rangle_N$ just introduces large random fluctuations around $N/2$, having little to do with the periodicity and the average behaviors of volatilities.

In this paper, we have found that the role of the memory in the game was responsible for generating a periodic response. Therefore, as long as we are interested in the volatility, the memory in the game is irrelevant. Either with memory or without memory, this model is nontrivial and should play an important role in the study of the complex adaptive systems.

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